ON THE COHOMOLOGY OF ARTIN GROUPS IN LOCAL SYSTEMS AND THE ASSOCIATED MILNOR FIBER

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Abstract. Let W be a finite irreducible Coxeter group and let $\boldsymbol{X}_W$ be the classifying space for G_W, the associated Artin group. If A is a commutative unitary ring, we consider the two local systems $\mathcal{L}_q$ and $\mathcal{L}_q'$ over $\boldsymbol{X}_W$, respectively over the modules $A[q, q^{-1}]$ and $A[[q, q^{-1}]]$, given by sending each standard generator of G_W into the automorphism given by the multiplication by q. We show that $H^*(\boldsymbol{X}_W, \mathcal{L}_q') = H^{*+1}(\boldsymbol{X}_W, \mathcal{L}_q)$ and we generalize this relation to a particular class of algebraic complexes. We remark that $H^*(\boldsymbol{X}_W, \mathcal{L}_q')$ is equal to the cohomology with trivial coefficients A of the Milnor fiber of the discriminant bundle of the associated reflection group.

Introduction

Let W be a finite irreducible Coxeter group (with Coxeter system (W, S)) and let G_W be the associated Artin group. Recall that if $W =< s, s \in S \mid (ss')^{m(s,s')} = e >$ is the standard presentation for the Coxeter group, then the standard presentation for G_W is given by

$$< g_s, s \in S \mid \overbrace{g_s g_{s'} g_s g_{s'} \cdots}^{m(s,s')\text{terms}} = \overbrace{g_{s'} g_s g_{s'} g_s \cdots}^{m(s,s')\text{terms}} \text{ for } s \neq s', m(s,s') \neq +\infty >$$

(see [2], [3] and [11]). We call $\boldsymbol{X}_W$ the classifying space for G_W. Let A be a commutative unitary ring; we consider a particular local system $\mathcal{L}_q$ over $\boldsymbol{X}_W$ with coefficients the ring $A[q, q^{-1}]$, where each standard generator of G_W acts as q-multiplication. Moreover let $\mathcal{L}_q'$ be the local system which is constructed in a similar way over the module $A[[q, q^{-1}]]$.

The cohomology groups $H^*(\boldsymbol{X}_W, \mathcal{L}_q')$ have an interesting geometrical interpretation, in fact they are equal to the cohomology groups (with trivial coefficients over the ring A) of the Milnor fiber $\boldsymbol{F}_W$ of the discriminant singularity associated to W (see section 2). From a straightforward application of the Shapiro Lemma ([4]) it is known that the homology groups $H_*(\boldsymbol{X}_W, \mathcal{L}_q)$ are equal to the homology groups of $\boldsymbol{F}_W$ with coefficients over the ring A (the argument is the same as that used in [6] for the homology of arrangements of hyperplanes).

The cohomology groups $H^*(\boldsymbol{X}_W, \mathcal{L}_q)$ and $H^*(\boldsymbol{X}_W, \mathcal{L}_q')$ can be computed by means of an algebraic complex described in [14]; in this paper we show (see equation (6)) that these groups coincide modulo an index shift, that is

$$H^*(\boldsymbol{X}_W, \mathcal{L}_q') = H^{*+1}(\boldsymbol{X}_W, \mathcal{L}_q).$$

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As a consequence we can use $\mathcal{L}_q$ to compute $H^*(\boldsymbol{F}_W, A)$. In the special case when $A = \mathbb{Q}$, and so the ring $A[q, q^{-1}]$ is a PID, the equality has already been observed ([7]) by Corrado De Concini. We also give a generalization of this fact, extending the result to a particular class of algebraic complexes including those described by Salvetti in [14].

In section 1 we give a precise formulation of the claim in an algebraic form and we give a proof of it by using spectral sequences. In section 2 we show how the algebraic result applies to the cohomology of Artin groups.

1. Main theorem

Remark 1.1. Let (C_1, d) be a graduated complex and let $C_3 \subset C_2 \subset C_1$ be inclusions of graduate complexes. Denote by $d_{ij} : C_i/C_j \to C_i/C_j$ the induced coboundary on the quotient complex ($1 \leq i < j \leq 3$). There is an obvious exact sequence of complexes:

$$0 \to C_2/C_3 \hookrightarrow C_1/C_3 \xrightarrow{\pi} C_1/C_2 \to 0$$

When d_{12} and d_{23} vanish (for example if the complexes are trivial in all degrees except exactly one) we get that $H^*(C_1/C_2) = C_1/C_2$ and $H^*(C_2/C_3) = C_2/C_3$, so the differential $H^*(C_1/C_2) \to H^*(C_2/C_3)$ of the long exact sequence associated to the above sequence gives a map

$$\overline{d} : C_1/C_2 \to C_2/C_3.$$

In the following we call this map *induced differential*.

Let A be a commutative unitary ring. In this section we indicate by $R = A[q, q^{-1}]$, the ring of Laurent polynomials with coefficients in A and by M the R-module $A[[q, q^{-1}]]$. Let (C^*, d^*) be a graduate cochain complex, with C^* an R-module and d^* an R-linear map. We give the following recursive definition:

Definition 1.2. *The complex (C^*, d^*) is called* well filtered *if C^* is a free finitely generated R-module, $C^* \neq R$ and moreover, if $C^* \neq 0$, the following conditions are satisfied:*

a) C^* is a filtered complex with a decreasing filtration F which is compatible with the coboundary map d^* and such that $F_0C = C^*$ and $F_{n+1}C = \{0\}$ for an integer $n > 0$;

b) $F_nC = (F_nC)^n \simeq F_{n-1}C/F_nC = (F_{n-1}C/F_nC)^{n-1} \simeq R$;

c) the induced differential $\overline{d} : F_{n-1}C/F_nC \to F_nC/F_{n+1}C$ (following from condition (b) and Remark 1.1) corresponds to the multiplication by a non-zero polynomial $p \in R$ with first and last non-zero coefficients invertible in A;

d) for all integer $i \neq n-1, n$ the induced complex $((F_iC/F_{i+1}C)^*, d_i^*)$ is a well filtered *complex.*

In the following when we consider a well filtered complex we always suppose to have also a filtration F as above. We write (C_M^*, d_M^*) for the complex $C^* \otimes_R M$ with the natural induced graduation and coboundary.

Theorem 1. *Let (C^*, d^*) be a well filtered complex. We have the following isomorphism:*

$$H^{*+1}(C^*) \simeq H^*(C_M^*).$$

In order to proof this fact we need two preliminary lemmas.

As a first step let us consider the natural inclusion of R-modules $R \hookrightarrow M$. We have the short exact sequence of R-modules:

$$0 \to R \hookrightarrow M \to M' \to 0$$

where $M' = M/R$. We indicate by C'^* the complex $C^* \otimes_R M'$ and we consider the complexes C^*, C^*_M, C'^*. In a similar way we have the following short exact sequence of R-modules:

$$0 \to C^* \overset{i}{\hookrightarrow} C^*_M \xrightarrow{\pi} C'^* \to 0.$$

Since the maps i and π commute with the coboundary maps, we actually have a short exact sequence of complexes. So we obtain the following long exact sequence:

$$(*) \qquad \begin{array}{l} \qquad\qquad\qquad \cdots \xrightarrow{\pi^*} H^{i-1}(C'^*) \xrightarrow{\delta^*} \\ \xrightarrow{\delta^*} H^i(C^*) \xrightarrow{i^*} H^i(C^*_M) \xrightarrow{\pi^*} H^i(C'^*) \xrightarrow{\delta^*} \\ \xrightarrow{\delta^*} H^{i+1}(C^*) \xrightarrow{i^*} \cdots \end{array}$$

Lemma 1.3. *Let (C^*, d^*) be a well filtered complex. With the notation given above we have:*

$$H^i(C'^*) \simeq H^i(C^*_M) \oplus H^i(C^*_M)$$

Proof. The R-module M' splits into the sum of two modules in the following way:

$$M' = M'_+ \oplus M'_-$$

where $M'_+ = M/(A[q][[q^{-1}]]), M'_- = M/(A[q^{-1}][[q]])$. In a similar way we get the splitting

$$C'^* = C'^*_+ \oplus C'^*_-.$$

Moreover C'^*_+ and C'^*_- are invariant for the coboundary induced by d^*, so the cohomology also splits:

$$H^*(C'^*) = H^*(C'^*_+) \oplus H^*(C'^*_-).$$

We want to show that the quotient projection $\pi_+ : C^*_M \to C'^*_+$ induces an isomorphism π^*_+ in cohomology. We will prove this by induction on the number of generators of C^* as a free R-module.

If $C^* = \{0\}$ the assertion is obvious. Suppose that C^* has m generators, with $m > 1$. Then the complexes $((F_iC/F_{i+1}C)^*, d^*_i)$ have a smaller number of generators and for $i \neq n-1, n$ they are well filtered. Therefore we can suppose by induction that the map π_{i+}, defined analogously to π_+, induces an isomorphism in cohomology for all the complexes $((F_iC/F_{i+1}C)^*, d^*_i)$, $i \neq n-1, n$, that is the map

$$\pi^*_{i+} : H^*((F_iC/F_{i+1}C)^* \otimes_R M) \to H^*((F_iC/F_{i+1}C)^* \otimes_R M'_+)$$

is an isomorphism for such i.

The filtration F on C^* induces filtrations on C^*_M and C'^*_+ in the following way: $F_iC_M = F_iC \otimes_R M$, $F_iC'_+ = F_iC \otimes_R M'_+$. We have the following natural isomorphisms:

$$(F_iC/F_{i+1}C)^* \otimes_R M \simeq (F_iC_M/F_{i+1}C_M)^*$$

$$(F_iC/F_{i+1}C)^* \otimes_R M'_+ \simeq (F_iC'_+/F_{i+1}C'_+)^*.$$

Through these isomorphisms the maps

$$(F_iC_M/F_{i+1}C_M)^* \to (F_iC'_+/F_{i+1}C'_+)^*$$

induced by π_+ correspond to π_{i+} and hence induce an isomorphism in cohomology for $i \neq n-1, n$.

Let us consider the spectral sequences $E_r^{i,j}$ and $\overline{E}_r^{i,j}$ associated to the complexes C^*_M and C'^*_+ with the respective filtrations. We write π_+^* also for the spectral sequences homomorphism induced by π_+. By the definition of the filtration F we have that $E_r^{i,j} = \overline{E}_r^{i,j} = 0$ if $i > n$ or if $i = n, n-1$ and $j \neq 0$. It is also clear that $E_1^{n-1,0} \simeq E_1^{n,0} = M$ and $\overline{E}_1^{n,0} \simeq \overline{E}_1^{n-1,0} = M'_+$. For $0 \leq i < n-1$ we get that $E_1^{i,j} \simeq H^{i+j}(F_iC^*_M/F_{i+1}C^*_M)$ and $\overline{E}_1^{i,j} \simeq H^{i+j}(F_iC'^*_+/F_{i+1}C'^*_+)$ therefore the inductive hypothesis gives that $E_1^{i,j} \simeq \overline{E}_1^{i,j}$ and the isomorphism between the terms of the spectral sequences is given by π_+^*. Now consider the maps $d_1^{n-1,0} : M \to M$ and $\overline{d}_1^{n-1,0} : M'_+ \to M'_+$. By condition (c) we have that these maps correspond to the multiplication by a non-zero polynomial $p = \sum_{i=s}^t b_iq^i$ with b_s, b_t invertible elements of the ring A. We can rewrite p as follows:

$$p = q^sb_s(1 + qp') = q^tb_t(1 + q^{-1}p'')$$

with $p' \in A[q]$, $p'' \in A[q^{-1}]$. Now we can look at these elements in M:

$$p_+^{-1} = q^{-s}b_s^{-1}\sum_{i=0}^{\infty}(-qp')^i$$

$$p_-^{-1} = q^{-t}b_t^{-1}\sum_{i=0}^{\infty}(-q^{-1}p'')^i.$$

Let $m \in M$, $m = \sum_{i\in\mathbb{Z}} a_iq^i$, we can write $m = m_+ + m_-$, with $m_+ = \sum_{i=0}^{\infty} a_iq^i$ and $m_- = m - m_+$. Notice that the products $p_+^{-1}m_+$ and $p_-^{-1}m_-$ are well defined and the following equality holds:

$$m = p(p_+^{-1}m_+ + p_-^{-1}m_-).$$

It turns out that the map $d_1^{n-1,0} : M \to M$ is surjective and the same holds, when passing to the quotient, for the map $\overline{d}_1^{n-1,0} : M_+ \to M_+$.

Let us suppose that an element $m = \sum_{i\in\mathbb{Z}} a_iq^i$ is in the kernel of $d_1^{n-1,0}$. This means that $pm = 0$, that is for all integers k we have:

$$\sum_{i=s}^{t} b_ia_{k-i} = 0$$

and so we obtain:

$$a_k = -b_s^{-1} \sum_{i=1}^{t-s} b_{s+i} a_{k-i} \tag{1}$$

$$a_k = -b_t^{-1} \sum_{i=1}^{t-s} b_{t-i} a_{k+i}. \tag{2}$$

Therefore if we know a sequence of $t-s$ consecutive coefficients of an element m sent to zero by the multiplication by p we can use (1) and (2) to calculate recursively all the other coefficients, determining m completely. So we find a bijection between $\ker d_1^{n-1,0}$ and $\ker \overline{d}_1^{n-1,0}$. In fact, if $m \in M$ is such that $pm = 0$, then trivially also $p[m]_+ = 0$ (we write $[m]_+$ for the equivalence class of m in M'_+). Conversely if $p[m]_+ = 0$ then we have $pm = z$, with $z \in A[q][[q^{-1}]]$, that is $z = \sum_{i \in \mathbb{Z}} v_i q^i$ with $v_i \in A$ and there exists an integer l such that $v_i = 0$ for all $i > l$. We can define recursively, for $j \geq 0$, the following elements:

$$\widetilde{a}[-1]_i = a_i$$

$$\widetilde{a}[j]_i = \begin{cases} \widetilde{a}[j-1]_i & \text{if} \quad i \neq l-t-j \\ -b_t^{-1} \sum_{k=1}^{t-s} b_{t-k} \widetilde{a}[j-1]_{i+k} & \text{if} \quad i = l-t-j \end{cases}$$

and

$$\widetilde{a}_i = \begin{cases} a_i & \text{if} \quad i > l-t \\ \widetilde{a}[l-t-i]_i & \text{if} \quad i \leq l-t \end{cases}$$

Notice that the coefficients v_i for $i > h$ depend only on the coefficients a_i for $i > h-t$, so if we write $\widetilde{m} = \sum_{i \in \mathbb{Z}} \widetilde{a}_i q^i$ we have that $p\widetilde{m} = 0$ and $[m]_+ = [\widetilde{m}]_+$.

To sum up we have that the map π_+^* gives an isomorphism between the terms $E_1^{i,j}$ and $\overline{E}_1^{i,j}$ for $i < n-1$ and between $\ker d_1^{n-1,0}$ and $\ker \overline{d}_1^{n-1,0}$. Moreover $E_2^{i,j} = \overline{E}_2^{i,j} = 0$ for $i = n-1$ and $j \neq 0$ and for $i > n-1$; π_+^* commutes with the differentials in the spectral sequences (i. e. $\pi_+^* d_i = \overline{d}_i \pi_+^*$). We remark that im $d_1^{n-2,0} \subset \ker d_1^{n-1,0}$ and im $\overline{d}_1^{n-2,0} \subset \ker \overline{d}_1^{n-1,0}$ and so π_+^* induces an isomorphism between im $d_1^{n-2,0}$ and im $\overline{d}_1^{n-2,0}$. This implies that π_+^* gives the isomorphisms $E_2^{n-2,0} \simeq \overline{E}_2^{n-2,0}$ and $E_2^{n-1,0} \simeq \overline{E}_2^{n-1,0}$. Then we have a complete isomorphism between E_2 and $\overline{E}_2$ and so between E_∞ and $\overline{E}_\infty$. It follows that π_+^* induces an isomorphism in cohomology.

It is clear that the same fact holds for the map $\pi_- : C_M^* \to C'^*_-$ and so Lemma is proved. $\square$

We write Φ for the isomorphism built in the proof of the previous Lemma.

Lemma 1.4. *In the exact sequence* $(*)$ *the map* π^* *composed with the isomorphism* Φ *corresponds to the diagonal map* Σ*:*

$$H^i(C_M^*) \xhookrightarrow{\Sigma} H^i(C_M^*) \oplus H^i(C_M^*).$$

Proof. It is enough to notice that, making the identification $H^*(C'^*) = H^*(C'^*_+) \oplus H^*(C'^*_-)$, we have that $\pi^* = (\pi_+^*, \pi_-^*)$ and so the statement follows immediately. $\square$

Proof (of Theorem 1). First of all we notice that, being π^* injective, i^* turns out to be the null map and δ^* is surjective. We call $p_1 : H^i(C^*_M) \oplus H^i(C^*_M) \to H^i(C^*_M)$ the projection on the first component, p_2 the projection on the second component and $i_1 : H^i(C^*_M) \hookrightarrow H^i(C^*_M) \oplus H^i(C^*_M)$ the inclusion defined by $i_1 : b \mapsto (b, 0)$. Finally we define $\alpha = \delta^* \circ \Phi^{-1} \circ i_1$. We have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & H^i(C^*_M) \arrow[r, hook, "\Sigma"] \arrow[d, equal] & H^i(C^*_M) \oplus H^i(C^*_M) \arrow[r, shift left, "p_1 - p_2"] \arrow[d, "\Phi", "\wr"'] & H^i(C^*_M) \arrow[r] \arrow[l, shift left, "i_1"] \arrow[d, "\alpha"] & 0 \\
0 \arrow[r] & H^i(C^*_M) \arrow[r, "\pi^*"] & H^i(C'^*_M) \arrow[r, "\delta^*"] & H^{i+1}(C^*) \arrow[r] & 0
\end{tikzcd}$$

Clearly both the lines are exact. We want to show that the diagram commutes. The commutativity for the first square follows by Lemma 1.4, so it remains to prove that the second square commutes. A pair $(a, b) \in H^i(C^*_M) \oplus H^i(C^*_M)$ is sent, by the multiplication by $p_1 - p_2$, into the element $a - b \in H^i(C^*_M)$. Then we have $i_1(a - b) = (a - b, 0)$ and the difference $(a, b) - (a - b, 0) = (b, b)$ is in the image of the map Σ. Therefore, because of the commutativity of the first square, the images of the pairs (a, b) and of $(a - b, 0)$ in $H^i(C'^*_M)$ are taken into the same element by the map δ^*. So we get the commutativity of the diagram. The Theorem follows from the five lemma. $\square$

2. Applications

Let us consider a finite set Γ endowed with a fixed total ordering. We will indicate by Δ a generic subset of Γ. We also set again $R = A[q, q^{-1}]$, with A a commutative unitary ring. For every pair (Δ, w) with $\Delta \subset \Gamma$, $w \in \Gamma \setminus \Delta$ we associate a polynomial $p_{\Delta,w}(q, q^{-1}) \in R \setminus \{0\}$ such that the first and the last non-zero coefficients are invertible in A. Let also suppose that for every pair (w, w') with $w \neq w'$ and $w, w' \in \Gamma \setminus \Delta$ the following equation holds:

$$p_{\Delta,w}(q, q^{-1}) p_{\Delta\cup\{w\},w'}(q, q^{-1}) + p_{\Delta,w'}(q, q^{-1}) p_{\Delta\cup\{w'\},w}(q, q^{-1}) = 0 \tag{3}$$

Then we can consider the complex (C^*_Γ, d^*) defined as follows:

$$C^*_\Gamma = \bigoplus_{\Delta \subset \Gamma} R.e_\Delta$$

$$d^* e_\Delta = \sum_{w \in \Gamma \setminus \Delta} p_{\Delta,w}(q, q^{-1}) e_{\Delta \cup \{w\}}.$$

We remark that the relation (3) gives $d^{*2} = 0$. We can also give a natural graduation to C^*_Γ by defining the degree of an element e_Δ as the cardinality of Δ, so we get a cochain complex.

Without loss of generality we can think $\Gamma = \{1, \ldots, n\}$. We introduce the following notation: indicate by Γ_i and Δ_i respectively the subsets $\{1, \ldots, n - i - 1\}$ and $\{n - i + 1, \ldots, n\}$. We can filter the complex C^*_Γ in the following way (see also [9]): let $F_i C_\Gamma$ be the subcomplex generated by the elements e_Δ, with $\Delta_i \subset \Delta$.

We have the following result:

Theorem 2. *With the filtration defined above the complex (C^*_Γ, d^*) is well filtered.*

Proof. We can prove this by induction on the cardinality of Γ. If Γ is empty the Theorem is obvious. Therefore let us suppose that the Theorem holds for all the complexes made up from a set with less than n elements and we prove it for a complex C^*_Γ, with $\Gamma = \{1, \ldots, n\}$.

It is straightforward to see that $F_0 C_\Gamma = C^*_\Gamma$ and $F_{n+1} C_\Gamma = \{0\}$. Moreover $F_n C_\Gamma$ and $F_{n-1} C_\Gamma / F_n C_\Gamma$ are generated respectively by the elements e_Γ and $e_{\Delta_{n-1}}$ and they are both isomorphic to R. The induced differential

$$\overline{d} : F_{n-1} C_\Gamma / F_n C_\Gamma \to F_n C_\Gamma / F_{n+1} C_\Gamma$$

corresponds to the multiplication by the polynomial $p_{\Delta_{n-1},1}(q, q^{-1})$.

Finally the complex $((F_i C_\Gamma / F_{i+1} C_\Gamma)^*, d_i^*)$ is isomorphic to the complex $C^*_{\Gamma_i}$, where the coboundary is defined by the polynomials

$$\overline{p}_{\Delta,j}(q, q^{-1}) := p_{\Delta \cup \Delta_i, j}(q, q^{-1}) \qquad \text{for } \Delta \subset \Gamma_i, j \in \Gamma_i \setminus \Delta$$

and so it is well filtered by induction. □

Now we apply last result and Theorem 1 to the cohomology with local coefficients of Artin groups. In [14] Salvetti proved that:

Theorem 3. *Let W be a Coxeter group with generating set Γ with a fixed total ordering and let G_W be the associated Artin group. Let R be a commutative ring with unit and let q be a unit in R and let M be an R-module. We write $W_\Delta(q)$ for the Poincaré polynomial of the subgroup of W generated by Δ, with $\Delta \subset \Gamma$. Let $\mathcal{L}_q = \mathcal{L}_q(X_W; M)$ be the local system over G_W with coefficients in M given by the map that sends every standard generator of G_W into the automorphism of M given by the multiplication by q. Then*

$$H^*(G_W; \mathcal{L}_q) \simeq H^*(C^*)$$

where

$$C^k = \{\sum a_\Delta e_\Delta \mid a_\Delta \in M, \Delta \subset \Gamma, |\Delta| = k\}$$

and the coboundary is given by

$$\delta^k(e_\Delta) = \sum_{j \in \Gamma \setminus \Delta} (-1)^{\sigma(j,\Delta)} \frac{W_{\Delta \cup \{j\}}(-q)}{W_\Delta(-q)} e_{\Delta \cup \{j\}}$$

where $\sigma(j, \Delta) = |\{i \in \Delta, i < j\}|$. □

Proposition 2.1. *Let $R = A[q, q^{-1}]$ and $M = R$. Then the complex C^* in Theorem 3 is well filtered.*

Proof. In fact the polynomial $W_\Gamma(q)$ divides $W_{\Gamma'}(q)$ when $\Gamma \subset \Gamma'$. Moreover the polynomials $W_\Gamma(q)$ are products of cyclotomic polynomials (see [1]), so they have first and last non-zero coefficients equal to 1. By using Theorem 2 we can easily see that C^* is well filtered. □

Now let W be a finite Coxeter group. We can think of W as a group generated by orthogonal reflections in a real vector space $\boldsymbol{V}$. Let $\mathfrak{H}$ be the arrangement of all the hyperplanes in $\boldsymbol{V}$ such that the associated orthogonal reflection is in W. We can consider the complexified space $\boldsymbol{V}_\mathbb{C}$ and the complexified arrangement $\mathfrak{H}_\mathbb{C}$. For every hyperplane $\boldsymbol{H} \in \mathfrak{H}_\mathbb{C}$ we chose a linear function $l_{\boldsymbol{H}}$ such that $\ker l_{\boldsymbol{H}} = \boldsymbol{H}$. The polynomial

$$\delta = \prod_{\boldsymbol{H} \in \mathfrak{H}_\mathbb{C}} l_{\boldsymbol{H}}^2$$

is called the discriminant of the arrangement and it is invariant with respect to the diagonal action of W on $\boldsymbol{V}_{\mathbb{C}}$. The space

$$\boldsymbol{X}_W = (\boldsymbol{V}_{\mathbb{C}} \setminus \cup_{\boldsymbol{H}\in\mathfrak{H}}\boldsymbol{H})/W$$

is a classifying space for the Artin group G_W (see [11]), and δ induces a fibering

$$\delta' : \boldsymbol{X}_W \to \mathbb{C}^*.$$

The fiber $\boldsymbol{F}_W = \delta^{-1}(1)$ is called the Milnor fiber of $\boldsymbol{D}_W = (\cup_{\boldsymbol{H}\in\mathfrak{H}}\boldsymbol{H})/W$. The associated homotopy exact sequence gives us that the $\boldsymbol{F}_W$ is a classifying space for the subgroup $H_W < G_W$, which is the kernel of the natural homomorphism

$$G_W \to \mathbb{Z}$$

defined by sending each standard generator to $+1$.

Now we set again $R = A[q, q^{-1}]$ and C^* and let $C^*_M = C^* \otimes M$ be the algebraic complexes defined as in Theorem 3, over R or $M = A[[q, q^{-1}]]$ respectively. Then (by definition) the following equality holds:

$$H^*(\boldsymbol{F}_W; A) = H^*(H_W; A) \tag{4}$$

and the Shapiro Lemma (see [4]) gives that

$$H^*(H_W; A) = H^*(G_W; Coind_{H_W}^{G_W} A) = H^*(G_W; M) = H^*(C^*_M) \tag{5}$$

where the action of G_W over M is given by sending each standard generator into the multiplication by q. From Theorem 1 and the remark following Theorem 3, we get that

$$H^*(G_W, M) = H^{*+1}(G_W, R) \tag{6}$$

Using equalities (4), (5) and (6) we get immediately the following result:

Theorem 4. *Let W be a finite irreducible Coxeter group and let*

$$\boldsymbol{F}_W \hookrightarrow \boldsymbol{X}_W \xrightarrow{\delta'} \mathbb{C}^*$$

be the fibration defined as before. Let $R = A[q, q^{-1}]$ be considered as a G_W-module with the action defined before. Then the following equality holds:

$$H^*(\boldsymbol{F}_W; A) = H^{*+1}(G_W; R)$$

□

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